

Deformed Statistics Free Energy Model for Source Separation using Unsupervised Learning

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Abstract—A generalized-statistics variational principle for source separation is formulated by recourse to Tsallis' entropy subjected to the additive duality and employing constraints described by normal averages. The variational principle is amalgamated with Hopfield-like learning rules resulting in an unsupervised learning model. The update rules are formulated with the aid of q -deformed calculus. Numerical examples exemplify the efficacy of this model.

I. INTRODUCTION

Recent studies have suggested that minimization of the Helmholtz free energy in statistical physics [1] plays a central role in understanding action, perception, and learning (see [2] and the references therein). In fact, it has been suggested that the principle of free energy minimization is even more fundamental than the redundancy reduction principle (also known as the principle of efficient coding) articulated by Barlow [3] and later formalized by Linsker as the Infomax principle [4]. Specifically, the principle of efficient coding states that the brain should optimize the mutual information between its sensory signals and some parsimonious neuronal representations. This is identical to optimizing the parameters of a generative model to maximize the accuracy of predictions, under complexity constraints. Both are mandated by the free-energy principle, which can be regarded as a probabilistic generalization of the Infomax principle.

The Infomax principle has been central to the development of independent component analysis (ICA) and the allied problem of blind source separation (BSS) [5]. Within the ICA/BSS context, very few models based on minimization of the free energy exist, the most prominent of them originated by Szu and co-workers (eg. see Refs. [6,7]) to achieve source separation in remote sensing (i.e. hyperspectral imaging (HSI)) using the maximum entropy principle. The ICA/BSS problem may be summarized in terms of the relation

$$\mathbf{A}\mathbf{s} = \mathbf{x}, \quad (1)$$

where \mathbf{s} is the *unknown* source vector to be extracted, \mathbf{A} is the *unknown* mixing matrix (also known as reflectance matrix or material abundance matrix in HSI), and \mathbf{x} is the *known* vector of observed data. The Helmholtz free energy is described within the framework of Boltzmann-Gibbs-Shannon (B-G-S) statistics as

$$F(T) = U - k_B T S, \quad (2)$$

where T is the thermodynamic temperature (or haemostatic temperature in the parlance of cybernetics), k_B the Boltzmann constant, U the internal energy, and S Shannon's entropy. A more principled and systematic manner in which to study free energy minimization within the context of the maximum entropy principle (MaxEnt) is by substituting the minimization of the Helmholtz free energy principle with the maximizing of the Massieu potential [8]

$$\Phi(\beta) = S - \beta U, \quad (3)$$

where $\beta = \frac{1}{k_B T}$ is the inverse thermodynamic temperature. The Massieu potential is the Legendre transform of the Helmholtz free energy, i.e.: $\Phi(\beta) = -\frac{F(T)}{T}$.

The generalized (also, interchangeably, nonadditive, deformed, or nonextensive) statistics of Tsallis has recently been the focus of much attention in statistical physics, complex systems, and allied disciplines [9]. Nonadditive statistics suitably generalizes the extensive, orthodox B-G-S one. The scope of Tsallis statistics has lately been extended to studies in lossy data compression in communication theory [10] and machine learning [11,12].

It is important to note that power law distributions like the q -Gaussian distribution cannot be accurately modeled within the B-G-S framework [9]. One of the most commonly encountered source of q -Gaussian distributions occurs in the process of normalization of measurement data using *Studentization* techniques [13]. q -Gaussian behavior is also exhibited by elliptically invariant data, which generalize spherically symmetric distributions. q -Gaussian's are also an excellent approximation to correlated Gaussian data, and other important and fundamental physical and biological processes (for example, see [14] and the references therein).

This paper intends to accomplish the following objectives:

- (i) to formulate and solve a variational principle for source separation using the maximum dual Tsallis entropy with constraints defined by normal averages expectations,
- (ii) to amalgamate the variational principle with Hopfield-like learning rules [15] to acquire information regarding unknown parameters via an unsupervised learning paradigm,
- (iii) to formulate a numerical framework for the generalized statistics unsupervised learning model and demon-

strate, with the aid of numerical examples for separation of independent sources (*endmembers*), the superiority of the generalized statistics source separation model *vis-à-vis* an equivalent B-G-S model for a single pixel.

It is important to note that by amalgamating the information-theoretic model with the Hopfield model, [A] acquires the role of the Associative Memory (AM) matrix. *Further, employing a Hopfield-like learning rule renders the model presented in this paper readily amenable to hardware implementation using Field Programmable Gate Arrays (FPGA's).*

The additive duality is a fundamental property in generalized statistics [9]. One implication of the additive duality is that it permits a deformed logarithm defined by a given nonadditivity parameter (say, q) to be inferred from its *dual deformed* logarithm parameterized by: $q^* = 2 - q$. This paper derives a variational principle for source separation using the dual Tsallis entropy using normal averages constraints. This approach has been previously utilized (for eg. Ref. [16]), and possess the property of seamlessly yielding a q^* -deformed exponential form on variational extremization.

An important issue to address concerns the manner in which expectation values are computed. Of the various forms in which expectations may be defined in nonextensive statistics has, only the linear constraints originally employed by Tsallis [9] (also known as *normal averages*) of the form: $\langle A \rangle = \sum_i p_i A_i$, has been found to be physically satisfactory and consistent with both the generalized H-theorem and the generalized *Stosszahlansatz* (molecular chaos hypothesis) [17, 18]. A re-formulation of the variational perturbation approximations in nonextensive statistical physics followed [18], via an application of q -deformed calculus [19]. Results from the study in Ref. [19] have been successfully utilized in Section IV of this paper.

This introductory Section is concluded by briefly describing the suitability of employing a generalized statistics model to study the source separation problem. First, in the case of remote sensing applications, and even more so in the case of HSI, the observed data are highly correlated, even in the case of a single pixel. Next, the observed data are required to be normalized (scaled). The *Studentization* process is one of the most prominent methods utilized to normalize the observed data [20,21]. Both these features lead to an excursion from the Gaussian framework (B-G-S statistics) and result in q -Gaussian pdf's characterized by the q -deformed exponential: $\exp_q(-x) = [1 - (1 - q)x]^{\frac{1}{1-q}}$, which maximizes the Tsallis entropy.

II. THEORETICAL PRELIMINARIES

The Section introduces the essential concepts around which this communication revolves. The Tsallis entropy is defined as [9]

$$S_q(X) = - \sum_x p(x)^q \ln_q p(x). \quad (4)$$

The q -deformed logarithm and the q -deformed exponential are defined as [9, 19]

$$\begin{aligned} \ln_q(x) &= \frac{x^{1-q} - 1}{1-q}, \\ \text{and,} \\ \exp_q(x) &= \begin{cases} [1 + (1-q)x]^{\frac{1}{1-q}}; & 1 + (1-q)x \geq 0, \\ 0; & \text{otherwise} \end{cases} \end{aligned} \quad (5)$$

Note that as $q \rightarrow 1$, (4) acquires the form of the equivalent B-G-S entropies. Likewise in (5), $\ln_q(x) \rightarrow \ln(x)$ and $\exp_q(x) \rightarrow \exp(x)$. The operations of q -deformed relations are governed by q -deformed algebra and q -deformed calculus [19]. Apart from providing an analogy to equivalent expressions derived from B-G-S statistics, q -deformed algebra and q -deformed calculus endow generalized statistics with a unique information geometric structure. The q -deformed addition \oplus_q and the q -deformed subtraction \ominus_q are defined as [19]

$$\begin{aligned} x \oplus_q y &= x + y + (1-q)xy, \\ \ominus_q y &= \frac{-y}{1+(1-q)y}; 1 + (1-q)y > 0 \\ \Rightarrow x \ominus_q y &= \frac{x-y}{1+(1-q)y} \end{aligned} \quad (6)$$

The q -deformed derivative, is defined as [19]

$$D_q^x F(x) = \lim_{y \rightarrow x} \frac{F(x) - F(y)}{x \ominus_q y} = [1 + (1-q)x] \frac{dF(x)}{dx} \quad (7)$$

As $q \rightarrow 1$, $D_q^x F(x) \rightarrow dF(x)/dx$, the Newtonian derivative. The Leibnitz rule for *deformed* derivatives [19] is

$$D_q^x [A(x) B(x)] = B(x) D_q^x A(x) + A(x) D_q^x B(x). \quad (8)$$

Re-parameterizing (5) via the *additive duality* [10]: $q^* = 2 - q$, yields the *dual deformed* logarithm and exponential

$$\ln_{q^*}(x) = -\ln_q\left(\frac{1}{x}\right), \text{ and, } \exp_{q^*}(x) = \frac{1}{\exp_q(-x)}. \quad (9)$$

The dual Tsallis entropy is defined by [10, 16]

$$S_{q^*}(X) = - \sum_x p(x) \ln_{q^*} p(x). \quad (10)$$

Here, $\ln_{q^*}(x) = \frac{x^{1-q^*} - 1}{1-q^*}$. The dual Tsallis entropies acquire a form similar to the B-G-S entropies, with $\ln_{q^*}(\bullet)$ replacing $\ln(\bullet)$.

III. VARIATIONAL PRINCIPLE

Consider the Lagrangian

$$\begin{aligned} \Phi_{q^*}[s_j] &= - \sum_j s_j \ln_{q^*} s_j - \sum_{i=1}^N \sum_{j=1}^N \lambda_i (A_{ij} s_j - x_i) \\ &+ \lambda_0 \left(\sum_{j=1}^N s_j - 1 \right), \end{aligned} \quad (11)$$

subject to the component-wise constraints

$$\sum_{j=1}^N s_j = 1, \text{ and } \sum_{j=1}^N A_{ij} s_j = x_i. \quad (12)$$

Clearly, the RHS of the Lagrangian (11) is the q^* -deformed Massieu potential: $\Phi_{q^*}[\lambda]$, subject to the normalization constraint on s_j . The variational extremization of (11), performed using the Ferri-Martinez-Plastino methodology [22], leads to

$$\begin{aligned} &\Rightarrow -\frac{(2-q^*)}{(1-q^*)} s_j^{1-q^*} - \sum_{i=1}^N \lambda_i A_{ij} + \lambda_0 = 0 \\ &\Rightarrow s_j = \left[\frac{(1-q^*)}{(q^*-2)} \left(-\lambda_0 + \sum_{i=1}^N \lambda_i A_{ij} \right) \right]^{\frac{1}{1-q^*}} \end{aligned} \quad (13)$$

Multiplying the second relation in (13) by s_j and summing over all j , yields after application of the normalization condition in (12)

$$-\frac{(2-q^*)}{(1-q^*)} \aleph_{q^*} - \sum_{j=1}^N \sum_{i=1}^N \lambda_i A_{ij} s_j = -\lambda_0, \quad (14)$$

where: $\aleph_{q^*} = \sum_{j=1}^N s_j^{2-q^*}$, and substituting (14) into the third relation in (13) yields

$$\begin{aligned} &s_j \\ &= \left[\aleph_{q^*} + (1-q^*) \sum_{j=1}^N \sum_{i=1}^N \tilde{\lambda}_i A_{ij} s_j - (1-q^*) \sum_{i=1}^N \tilde{\lambda}_i A_{ij} \right]^{\frac{1}{1-q^*}} \\ &\tilde{\lambda}_i = \frac{\lambda_i}{(2-q^*)}. \end{aligned} \quad (15)$$

Eq. (15) yields after some algebra

$$s_j = \frac{\exp_{q^*} \left(- \sum_{i=1}^N \tilde{\lambda}_i^* A_{ij} \right)}{\left(\aleph_{q^*} + (1-q^*) \sum_{j=1}^N \sum_{i=1}^N \tilde{\lambda}_i A_{ij} s_j \right)^{\frac{1}{q^*-1}}}, \quad (16)$$

where

$$\begin{aligned} \tilde{\lambda}_i^* &= \frac{\tilde{\lambda}_i}{\aleph_{q^*} + (1-q^*) \sum_{j=1}^N \sum_{i=1}^N \tilde{\lambda}_i A_{ij} s_j}, \\ \text{and,} \\ \left(\aleph_{q^*} + (1-q^*) \sum_{j=1}^N \sum_{i=1}^N \tilde{\lambda}_i A_{ij} s_j \right)^{\frac{1}{q^*-1}} &= \tilde{Z}_{q^*}. \end{aligned} \quad (17)$$

Here \tilde{Z}_{q^*} is the canonical partition function, where: $\tilde{Z}_{q^*} = \sum_{j=1}^N \exp_{q^*} \left(- \sum_{i=1}^N \tilde{\lambda}_i^* A_{ij} \right)$. The dual Tsallis entropy takes the form

$$\begin{aligned} S_{q^*}[s] &= \frac{\aleph_{q^*}-1}{(q^*-1)}; \sum_{j=1}^N s_j = 1 \\ \Rightarrow \aleph_{q^*} &= 1 + (q^* - 1) S_{q^*}[s] \end{aligned} \quad (18)$$

Substituting now (18) into the expression for: \tilde{Z}_{q^*} in (17) results in

$$-\ln_{q^*} \left(\frac{1}{\tilde{Z}_{q^*}} \right) = S_{q^*}[s] - \sum_{j=1}^N \sum_{i=1}^N \tilde{\lambda}_i A_{ij} s_j = \Phi_{q^*}[\tilde{\lambda}]. \quad (19)$$

Clearly, $\Phi_{q^*}[\tilde{\lambda}]$ in (19) is a q^* -deformed Massieu potential. By substituting (18) into (14) we arrive at

$$\begin{aligned} S_{q^*}[s] - \sum_{j=1}^N \sum_{i=1}^N \tilde{\lambda}_i A_{ij} s_j &= -\tilde{\lambda}_0 + \frac{1}{(1-q^*)} = \hat{\lambda}_0; \\ \tilde{\lambda}_0 &= \frac{\lambda_0}{(2-q^*)}. \end{aligned} \quad (20)$$

Again, $\hat{\lambda}_0$ in (20) is a q^* -deformed Massieu potential: $\Phi_{q^*}[\tilde{\lambda}]$. We wish to relate $\hat{\lambda}_0$ and \tilde{Z}_{q^*} . To this end, comparison of (19) and (20) yields

$$\begin{aligned} \hat{\lambda}_0 &= -\tilde{\lambda}_0 + \frac{1}{(1-q^*)} = -\frac{\tilde{Z}_{q^*}^{q^*-1}}{(1-q^*)} + \frac{1}{(1-q^*)} \\ \Rightarrow \tilde{Z}_{q^*} &= \left[(1-q^*) \tilde{\lambda}_0 \right]^{\frac{1}{q^*-1}}; \tilde{\lambda}_0 = \frac{\lambda_0}{(2-q^*)}, \end{aligned} \quad (21)$$

so that, by substituting (18) into (15) and then invoking (20) we get

$$\begin{aligned} s_j &= \left[1 - (1-q^*) \left(\sum_{i=1}^N \tilde{\lambda}_i A_{ij} + \hat{\lambda}_0 \right) \right]^{\frac{1}{1-q^*}}; \\ \hat{\lambda}_0 &= -\tilde{\lambda}_0 + \frac{1}{(1-q^*)}. \end{aligned} \quad (22)$$

Here, (22) is re-defined with the aid of (20) as

$$\begin{aligned} s_j &= \frac{\left[1 - (1-q^*) \sum_{i=1}^N \tilde{\lambda}_i^* A_{ij} \right]^{\frac{1}{1-q^*}}}{\left[1 - (1-q^*) \tilde{\lambda}_0 \right]^{\frac{1}{q^*-1}}} = \frac{\left[1 - (1-q^*) \sum_{i=1}^N \tilde{\lambda}_i^* A_{ij} \right]^{\frac{1}{1-q^*}}}{\tilde{Z}_{q^*}}; \\ \text{where} \\ \tilde{\lambda}_i^* &= \frac{\tilde{\lambda}_i}{1 - (1-q^*) \tilde{\lambda}_0}, \tilde{Z}_{q^*} = \sum_{j=1}^N \left[1 - (1-q^*) \sum_{i=1}^N \tilde{\lambda}_i^* A_{ij} \right]^{\frac{1}{1-q^*}}. \end{aligned} \quad (23)$$

With the aid of (21), (22) is re-cast in the form

$$\begin{aligned} s_j &= \frac{\exp_{q^*} \left(- \sum_{i=1}^N \tilde{\lambda}_i^* A_{ij} \right)}{\left[(1-q^*) \tilde{\lambda}_0 \right]^{\frac{1}{q^*-1}}}; \\ \text{where, } \tilde{\lambda}_i &= \frac{\lambda_i}{(2-q^*)}, \tilde{\lambda}_0 = \frac{\lambda_0}{(2-q^*)}, \tilde{\lambda}_i^* = \frac{\tilde{\lambda}_i}{[(1-q^*) \tilde{\lambda}_0]}. \end{aligned} \quad (24)$$

Finally, invoking the normalization of s_j , (24) yields

$$\left[(1-q^*) \tilde{\lambda}_0 \right]^{\frac{1}{q^*-1}} = \sum_{j=1}^N \left[1 - (1-q^*) \sum_{i=1}^N \tilde{\lambda}_i^* A_{ij} \right]^{\frac{1}{1-q^*}}. \quad (25)$$

Note the *self-referential* nature of (23) in the sense that: $\tilde{\lambda}_i^*$ (defined in (20) and (23) is a function of $\tilde{\lambda}_0$. The Lagrange multiplier $\tilde{\lambda}_i^*$ is henceforth defined in this paper as the *dual normalized Lagrange force multiplier*.

IV. UNSUPERVISED LEARNING RULES

The process of unsupervised learning is amalgamated to the above information theoretic structure via a Hopfield-like learning rule to update the AM matrix $[\mathbf{A}]$ in the case of a

perturbation Δx_j of the observed data

$$\begin{aligned} \frac{dx_j}{dt} &= \frac{\partial \Phi_{q^*}^*[s_j]}{\partial s_j} \\ &= -\frac{1-(1-q^*)\tilde{\lambda}}{(1-q^*)\lambda_0} - \frac{\ln_{q^*} s_j}{\lambda_0} - (1-q^*) \sum_{i=1}^N \tilde{\lambda}_i^* A_{ij} \\ &\Rightarrow \Delta x_j \\ &= -\left[\frac{1-(1-q^*)\tilde{\lambda}_0}{(1-q^*)\lambda_0} + \frac{\ln_{q^*} s_j}{\lambda_0} + (1-q^*) \sum_{i=1}^N \tilde{\lambda}_i^* A_{ij} \right] \Delta t; \\ \text{where, } \tilde{\Phi}_{q^*}^*[s_j] &= \frac{\Phi_{q^*}[s_j]}{(2-q^*)\lambda_0}, \end{aligned} \quad (26)$$

which is obtained from the first relation in (13) and (24). Gradient ascent along with (24) originates the second learning rule

$$\begin{aligned} \frac{dx_j}{dt} &= \frac{\partial \Phi_{q^*}^*[s_j]}{\partial A_{ij}} = -\tilde{\lambda}_i^* s_j \Rightarrow \Delta x_j = -(\tilde{\lambda}_i^* s_j) \Delta t; \\ \text{where, } \Phi_{q^*}^*[s_j] &= \frac{\Phi_{q^*}[s_j]}{(1-q^*)\lambda_0}. \end{aligned} \quad (27)$$

In (26) and (27), $\Phi_{q^*}[s_j]$ is the LHS of the Lagrangian (11).

Now, a *critical* update rule is that for the change in the *dual normalized Lagrange force multipliers* $\tilde{\lambda}_i^*$ resulting from a perturbation Δx_j in the observed data. Usually (as stated within the context of the B-G-S framework), such an update would entail a Taylor-expansion yielding up to the first order: $\Delta x_j = \sum_{k=1}^N \frac{\partial x_j}{\partial \tilde{\lambda}_k^*} \Delta \tilde{\lambda}_k^*$. Such an analysis is valid only for distributions characterized by the regular exponential $\exp(-x)$. For probability distributions characterized by q -deformed exponentials, i.e., the ones we face here, such a perturbation treatment would lead to un-physical results [18].

Thus, following the prescription given in Ref. [18], for a function: $F(\tau) = \sum_n F(\tau_n)$ the chain rule yields:

$\frac{dF(\tau)}{d\tilde{\lambda}_k^*} = \frac{dF(\tau)}{d\tau} \frac{d\tau}{d\tilde{\lambda}_k^*}$. Thus, replacing the Newtonian derivative: $\frac{dF(\tau)}{d\tau}$ by the q^* -deformed one defined by (7) (see Ref. [19]): $D_{q^*}^\tau F(\tau) = [1 + (1-q^*)\tau] \frac{dF(\tau)}{d\tau}$ and defining: $D_{q^*}^\tau F(\tau) \frac{d\tau}{d\tilde{\lambda}_k^*} = \delta_{q^*,\tau} F(\tau)$ as well, facilitates the desired transformation: $\frac{dF(\tau)}{d\tilde{\lambda}_k^*} \rightarrow \delta_{q^*,\tau} F(\tau)$. Consequently, the update rule for $\tilde{\lambda}_k^*$ is re-formulated via q -deformed calculus in the fashion

$$\Delta x_j = \sum_{k=1}^N \left[D_{q^*}^\tau \sum_{i=1}^N A_{ji} s_i \right] \Delta \tilde{\lambda}_k^* = \sum_{k=1}^N \left[\sum_{i=1}^N D_{q^*}^\tau A_{ji} s_i \right] \Delta \tilde{\lambda}_k^*. \quad (28)$$

Additionally, setting: $-A_{ik} \tilde{\lambda}_k^* = \tau$ in (23) leads to

$$s_j = \frac{[1 + (1-q^*)\tau]^{\frac{1}{1-q^*}}}{\tilde{Z}_{q^*}}. \quad (29)$$

Employing at this stage the Leibnitz rule for q^* -deformed derivatives (and replacing q by q^* in (8)), the term within square parenthesis RHS in (28) yields

$$\begin{aligned} \sum_{i=1}^N D_{q^*}^\tau A_{ji} s_i &= \sum_{i=1}^N \left\{ \frac{A_{ji}}{\tilde{Z}_{q^*}} D_{q^*}^\tau [1 + (1-q^*)\tau]^{\frac{1}{1-q^*}} \right. \\ &\quad \left. + A_{ji} [1 + (1-q^*)\tau]^{\frac{1}{1-q^*}} D_{q^*}^\tau \left(\frac{1}{\tilde{Z}_{q^*}} \right) \right\}, \end{aligned} \quad (30)$$

a relation that, after expansion turns into

$$\begin{aligned} &\sum_{i=1}^N D_{q^*}^\tau A_{ji} s_i \\ &= \sum_{i=1}^N \left\{ \frac{A_{ji}}{\tilde{Z}_{q^*}} [1 + (1-q^*)\tau] \frac{\partial \tau}{\partial \tilde{\lambda}_k^*} \frac{\partial}{\partial \tau} [1 + (1-q^*)\tau]^{\frac{1}{1-q^*}} \right. \\ &\quad \left. + A_{ji} [1 + (1-q^*)\tau]^{\frac{1}{1-q^*}} D_{q^*}^\tau \left(\frac{1}{\tilde{Z}_{q^*}} \right) \right\} \\ &= \sum_{i=1}^N \left\{ -\frac{A_{ji}}{\tilde{Z}_{q^*}} [1 + (1-q^*)\tau]^{\frac{1}{1-q^*}} A_{ik} \right. \\ &\quad \left. - A_{ji} [1 + (1-q^*)\tau]^{\frac{1}{1-q^*}} [1 + (1-q^*)\tau] \frac{\partial \tau}{\partial \tilde{\lambda}_k^*} \tilde{Z}_{q^*}^{-2} \frac{\partial \tilde{Z}_{q^*}}{\partial \tau} \right\} \\ &= -\sum_{i=1}^N A_{ji} s_i A_{ik} \\ &\quad + \sum_{i=1}^N A_{ji} \frac{[1 + (1-q^*)\tau]^{\frac{1}{1-q^*}}}{\tilde{Z}_{q^*}} \sum_{k=1}^N \frac{A_{ik}}{\tilde{Z}_{q^*}} [1 + (1-q^*)\tau]^{\frac{1}{1-q^*}} \\ &= -\sum_{i=1}^N A_{ji} s_i A_{ik} + x_j x_k. \end{aligned} \quad (31)$$

Finally, the update rule for $\tilde{\lambda}_k^*$ with respect to Δx_j adopts the appearance

$$\Delta x_j = \sum_{k=1}^N \left(x_j x_k - \sum_{i=1}^N A_{ji} s_i A_{ik} \right) \Delta \tilde{\lambda}_k^*. \quad (32)$$

V. NUMERICAL COMPUTATIONS

The procedure for our double recursion problem is summarized in the pseudo-code below

Algorithm 1 Generalized Statistics Source Separation Model

- (1.) **Input:** (i). Observed data: \mathbf{x} , (ii). Trial values of dual normalized Lagrange force multipliers: $\tilde{\lambda}^*$, (iii). Dual nonadditive parameter: q^* .
- (2.) **Initialization:**
Obtain $A_{ij}^{(0)}$ from: $A_{ij}^{(0)} = x_i \sigma_{q^*}(x_j) + 50\%$ random noise to break any rank-1 singularity. The q^* -deformed sigmoid logistic function is: $\sigma_{q^*}(x_j) = \frac{1}{1 + \exp_{q^*}(-x_j)}$.
- (3.) **First Recursion**
(i) Compute: $\tilde{Z}_{q^*}^{(0)}$ from (23),
(ii) Compute: $\tilde{\lambda}_i^{(0)}$ from (21),
(iii) Compute: $s_j^{(0)}$, $\tilde{\lambda}_i^{(0)}$, and $\tilde{\lambda}_0^{(0)}$ from (23)/(24),
(iv) Compute: $x_j^{(0)}$ from (5), thus: $\Delta x_j^{(0)} = x_j^{Known} - x_j^{(0)}$,
(v) Compute $\Delta \tilde{\lambda}_k^{*(0)}$ by inverting (32),
(vi) Compute next estimate: $\tilde{\lambda}_k^{*(1)} = \tilde{\lambda}_k^{*(0)} + \Delta \tilde{\lambda}_k^{*(0)}$.
- (4.) **Second Recursion**
(vii) Compute improved estimate of : $A_{ij}^{(1)}$ from (26) by setting $\Delta t = 1$ and solving : $\Delta x_j = -\left[\frac{1-(1-q^*)\tilde{\lambda}_0^{(0)}}{(1-q^*)\tilde{\lambda}_0^{(0)}} + \frac{\ln_{q^*} s_j^{(0)}}{\tilde{\lambda}_0^{(0)}} + (1-q^*) \sum_{i=1}^N \tilde{\lambda}_i^{*(1)} A_{ij}^{(1)} \right]$.
- (5.) Go to (3.)

Following the procedure outlined in the above pseudo-code, values of $\tilde{\lambda}^* = [0.6228, 0.6337, 0.4577, 0.1095, 0.7252, 0.01752, 0.4128]$ and $\mathbf{x} = [0.5382, 0.1023, 0.6404, 0.4358, 0.0278, 0.2425, 0.3299]$

are provided. These values are the same as those in Ref. [7] and constitute experimentally obtained Landsat data for a single pixel. The difference between the generalized statistics model presented in this paper and the B-G-S model of [6,7] lies in the fact that the former has initial inputs of $\tilde{\lambda}_i^*$'s, whereas the latter merely has initial inputs of λ 's (a far simpler case). *The self-referentiality in (23) mandates use of $\tilde{\lambda}_i^*$'s as the primary operational Lagrange multiplier.* Note that the correlation coefficient of x^{Known} is unity, a signature of highly correlated data. A value of $q^* = 0.75$ is chosen. Figure 1 and Figure 2 depict, vs. the number of iterations, the source separation for the generalized statistics model and for the B-G-S model, respectively. Values of \mathbf{x} are denoted by "o"s. It is readily appreciated that the generalized statistics exhibits a more pronounced source separation than the B-G-S model. Owing to the highly correlated nature of the observed data, such results are to be expected.

VI. SUMMARY AND DISCUSSION

A generalized statistics model for source separation that employs an unsupervised learning paradigm has been presented in this communication. This model is shown to exhibit superior separation performance as compared to an equivalent model derived within the B-G-S framework. Our encouraging results should inspire future work studies on the implications of first-order and second-order phase transitions of the Massieu potential. One would wish for a self-consistent scheme enabling one to obtain self-consistent values of Lagrange multipliers based on the principle of phase transitions and symmetry breaking.

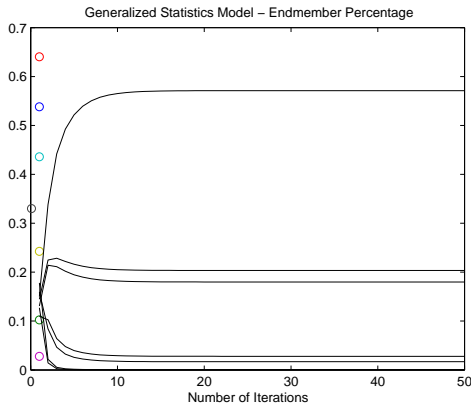


Fig. 1. Source separation for generalized statistics model

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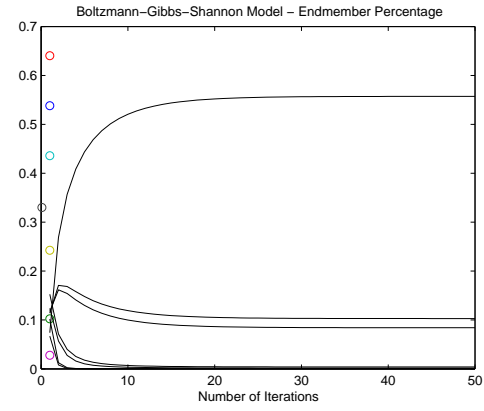


Fig. 2. Source separation for Boltzmann-Gibbs-Shannon model

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